Sinusoidal Stady-State Response of Discrete-Time Systems

Processament Digital del senyal — Enginyeria de Sistemes TIC

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1 Motivation

This document covers the computation of the sinusoidal steady-state response (**SSR**) of discrete-time systems (**DTS**).

First we compute the sinusoidal SSR for a particular filter, the **moving average filter**, using the **Z-transform**. This is done starting the computation of the general response but, before completing the computation of this general response, we discard the **natural response** and only compute to the end the **forced response**.

Next we focus on the way this response has been computed, specially on the role that the **transfer function** of the system $H(z)$ has in the final result, in order to obtain a general method to easily compute the sinusoidal SSR of DTS.

2 A particular case: the moving average filter

On the one hand, the moving average filter is characterized by the following relation between the input $x(n)$ and the output $y(n)$:

$$y(n) = \frac{1}{m+1}\sum_{k=0}^{k=m} x(n-k). \tag{2.0.1}$$

Note that the number of samples used to compute the output is $m+1$. The impulse response of this system can easily be obtained as

$$h(n) = \frac{1}{m+1} \sum_{k=0}^{k=m} \delta(n-k). \tag{2.0.2}$$

Being the moving average filter a LTI system we use the Z-transform to obtain its transfer function

$$H(z) = \frac{1}{m+1} \sum_{k=0}^{k=m} z^{-k} = \frac{1}{m+1} \frac{1-z^{-(m+1)}}{1-z^{-1}}. \tag{2.0.3}$$

On the other hand, an analog sinusoidal signal in the time domain is characterized by its frequency, its amplitude and its phase. We will consider the analog signal

$$x_a(t) = \sin\left(2\pi \mathrm{F}_a t\right) u(t) \tag{2.0.4}$$

of analog frequency F_a. Sampling $x_a(t)$ at frequency F_s we obtain the discrete-time signal

$$x(n) = \sin\left(\omega_0 n\right) u(n), \tag{2.0.5}$$

where $\omega_0 = 2\pi f = 2\pi \frac{F_a}{F_s}$. The signal (2.0.5) has amplitude one. Being the moving average filter a linear filter, all the results are obtained without loss of generality. The signal (2.0.5) has the phase of a sinus. Being the moving average filter an invariant time filter, the SSR results are also obtained without loss of generality [1].

Considering the Z-transform of (2.0.5)

$$X(z) = \frac{\sin\left(\omega_0\right) z^{-1}}{1 - 2\cos\left(\omega_0\right) z^{-1} + z^{-2}} \tag{2.0.6}$$

as the input to the moving average filter, we can write its Z-transform response as

$$Y(z) = H(z)X(z) = \frac{\sin\left(\omega_0\right)}{m+1} \frac{(1-z^{-(m+1)})z^{-1}}{(1-z^{-1})(1-2\cos\left(\omega_0\right) z^{-1} + z^{-2})}. \tag{2.0.7}$$

A (strictly) proper transfer function is a transfer function in which the degree M of the numerator is less than the degree N of the denominator [2]. Otherwise, we have an improper function that can be written as

$$\begin{aligned} Y_I(z) = \frac{N_M(z)}{D_N(z)} &= \frac{N_{N-1}(z)}{D_N(z)} + \sum_{k=0}^{k=M-N} c_k z^{-k} \\ &= Y_P(z) + \sum_{k=0}^{k=M-N} c_k z^{-k}. \end{aligned} \tag{2.0.8}$$

The inverse Z-transform (ZT^{-1}) of the right side of (2.0.8) is easily obtained as

$$\sum_{k=0}^{k=M-N} c_k z^{-k} \xrightarrow{ZT^{-1}} \sum_{k=0}^{k=M-N} c_k \delta(n-k). \tag{2.0.9}$$

[1]However, this is not true for transient results.
[2]Numerator and denominator are polynomials in z^{-1}.

Our particular response (2.0.7) is an improper function with $N = 3$ and $M = m + 2$ [3].

Considering again (2.0.8) and (2.0.9) we can make a first draft of the time domain response:

$$Y_I(z) = \frac{N_{N-1}(z)}{D_N(z)} + \sum_{k=0}^{k=m-1} c_k z^{-k} \xrightarrow{ZT^{-1}} y_I(n) = y_P(n) + \sum_{k=0}^{k=m-1} c_k \delta(n-k). \tag{2.0.10}$$

The addition that appears on the right part of the equation lasts, in the time domain, just m samples, from $n = 0$ to $n = m - 1$. As we are interested in the sinusoidal SSR, we will ignore this transient response and we will focus on the time response of

$$Yp(z) = \frac{N_{N-1}(z)}{D_N(z)} \xrightarrow{ZT^{-1}} y_P(n). \tag{2.0.11}$$

The ZT^{-1} in (2.0.11) can be obtained with the same procedure used to compute the inverse Laplace Transform: partial-fraction expansion. A proper transfer function can be expanded in the form

$$Y_p(z) = \sum_{k=1}^{k=N} \frac{A_k}{1 - p_k z^{-1}}, \tag{2.0.12}$$

where p_k is each one of the poles of $Y_p(z)$ [4].

Looking at the denominator of (2.0.7),

$$\begin{aligned} D_N(z) &= (1 - z^{-1})(1 - 2\cos(\omega_0)\, z^{-1} + z^{-2}) \\ &= (1 - z^{-1})(1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1}), \end{aligned} \tag{2.0.13}$$

it is straightforward that the poles in (2.0.12) are $p_1 = 1$, $p_2 = e^{j\omega_0}$ and $p_3 = e^{-j\omega_0}$, and so, we can write (2.0.13) as

$$D_N(z) = (1 - p_1 z^{-1})(1 - p_2 z^{-1})(1 - p_3 z^{-1}). \tag{2.0.14}$$

2.1 Computation of the coefficients

Once the coefficients A_k in (2.0.12) are computed, the inversion of the Z-transform of $Y_p(z)$ is straightforward:

$$y_p(n) = \sum_{k=1}^{n} A_k\, (p_k)^n\, u(n). \tag{2.1.1}$$

2.1.1 Computation of A_1

But first we must compute each A_k. We start with A_1. A common approach is the following [5]

$$\begin{aligned} Y(z)(1 - z^{-1})\Big|_{z=1} = \Bigg(&\frac{A_1}{1 - z^{-1}}(1 - z^{-1}) \\ &+ \frac{A_2}{1 - e^{j\omega_0} z^{-1}}(1 - z^{-1}) + \frac{A_3}{1 - e^{-j\omega_0} z^{-1}}(1 - z^{-1}) \\ &+ \left(\sum_{k=0}^{k=m-1} c_k z^{-k} \right)(1 - z^{-1}) \Bigg)\Bigg|_{z=1} \end{aligned} \tag{2.1.2}$$

[3] Except for the case $m = 0$ that means $H(z) = 1$ and, as a consequence, $Y(z) = X(z)$.

[4] We have assumed that each p_k has multiplicity one.

[5] Note that although we could use $Y_P(z)$ instead of $Y(z)$, the use of $Y(z)$ needs no manipulation.

$$Y(z)(1-z^{-1})\Big|_{z=1} = A_1 + \left(\frac{A_2}{1-e^{j\omega_0}z^{-1}}(1-z^{-1}) + \frac{A_3}{1-e^{-j\omega_0}z^{-1}}(1-z^{-1}) + \left(\sum_{k=0}^{k=m-1} c_k z^{-k}\right)(1-z^{-1})\right)\Bigg|_{z=1} \tag{2.1.3}$$

$$Y(z)(1-z^{-1})\Big|_{z=1} = A_1. \tag{2.1.4}$$

Using (2.0.7) we write

$$A_1 = \frac{\sin(\omega_0)}{m+1}\frac{(1-z^{-(m+1)})z^{-1}}{(1-z^{-1})(1-2\cos(\omega_0)z^{-1}+z^{-2})}(1-z^{-1})\Bigg|_{z=1} \tag{2.1.5}$$

$$A_1 = \frac{\sin(\omega_0)}{m+1}\frac{(1-z^{-(m+1)})z^{-1}}{(1-2\cos(\omega_0)z^{-1}+z^{-2})}\Bigg|_{z=1} = 0. \tag{2.1.6}$$

Going backward, we see that the pole $p_1 = 1$ in the denominator of $H(z)$ in (2.0.3) is canceled by the zero $z_1 = 1$ in the numerator. The pole p_1 appears in the denominator because we have used a compact formulation to write $H(z)$. If we had previously made this cancellation, there will be no pole p_1. This explanation is coherent with the computed result $A_1 = 0$.

2.1.2 Computation of A_2

The computation of A_2 follows the same approach.

$$Y(z)(1-e^{j\omega_0}z^{-1})\Big|_{z=e^{j\omega_0}} = \left(\frac{A_1}{1-z^{-1}}(1-e^{j\omega_0}z^{-1}) + \frac{A_2}{1-e^{j\omega_0}z^{-1}}(1-e^{j\omega_0}z^{-1}) + \frac{A_3}{1-e^{-j\omega_0}z^{-1}}(1-e^{j\omega_0}z^{-1}) + \left(\sum_{k=0}^{k=m-1} c_k z^{-k}\right)(1-e^{j\omega_0}z^{-1})\right)\Bigg|_{z=e^{j\omega_0}} \tag{2.1.7}$$

$$Y(z)(1-e^{j\omega_0}z^{-1})\Big|_{z=e^{j\omega_0}} = A_2 + \left(\frac{A_1}{1-z^{-1}}(1-e^{j\omega_0}z^{-1}) + \frac{A_3}{1-e^{-j\omega_0}z^{-1}}(1-e^{j\omega_0}z^{-1}) + \left(\sum_{k=0}^{k=m-1} c_k z^{-k}\right)(1-e^{j\omega_0}z^{-1})\right)\Bigg|_{z=e^{j\omega_0}} \tag{2.1.8}$$

$$Y(z)(1-e^{j\omega_0}z^{-1})\Big|_{z=e^{j\omega_0}} = A_2. \tag{2.1.9}$$

Using (2.0.7) again, and also (2.0.13), we write

$$A_2 = \frac{\sin(\omega_0)}{m+1} \frac{(1-z^{-(m+1)})z^{-1}}{(1-z^{-1})(1-e^{j\omega_0}z^{-1})(1-e^{-j\omega_0}z^{-1})}(1-e^{j\omega_0}z^{-1})\Bigg|_{z=e^{j\omega_0}} \tag{2.1.10}$$

$$A_2 = \frac{\sin(\omega_0)}{m+1} \frac{(1-z^{-(m+1)})z^{-1}}{(1-z^{-1})(1-e^{-j\omega_0}z^{-1})}\Bigg|_{z=e^{j\omega_0}} \tag{2.1.11}$$

$$A_2 = \frac{\sin(\omega_0)}{m+1} \frac{(1-e^{-j(m+1)\omega_0})e^{-j\omega_0}}{(1-e^{-j\omega_0})(1-e^{-j2\omega_0})} \tag{2.1.12}$$

Some transformations can be made on (2.1.12) considering that the three terms like

$$1-e^{-jk\omega_0}, \tag{2.1.13}$$

with $k = \{1, 2, m+1\}$, can be written as

$$\begin{aligned} 1-e^{-jk\omega_0} &= e^{-j\frac{k}{2}\omega_0}e^{j\frac{k}{2}\omega_0}\left(1-e^{-jk\omega_0}\right) \\ &= e^{-j\frac{k}{2}\omega_0}\left(e^{j\frac{k}{2}\omega_0}-e^{-j\frac{k}{2}\omega_0}\right) \\ &= 2je^{-j\frac{k}{2}\omega_0}\sin\left(\omega_0\frac{k}{2}\right). \end{aligned} \tag{2.1.14}$$

So, substituting (2.1.14) in (2.1.12) we obtain

$$\begin{aligned} A_2 &= \frac{\sin(\omega_0)}{m+1} \frac{2je^{-j\frac{m+1}{2}\omega_0}\sin\left(\omega_0\frac{m+1}{2}\right)e^{-j\omega_0}}{2je^{-j\frac{1}{2}\omega_0}\sin\left(\omega_0\frac{1}{2}\right)2je^{-j\omega_0}\sin(\omega_0)} \\ &= \frac{\sin\left(\omega_0\frac{m+1}{2}\right)}{2(m+1)\sin\left(\omega_0\frac{1}{2}\right)}\frac{e^{-j\frac{m}{2}\omega_0}}{j} \\ &= \frac{\sin\left(\omega_0\frac{m+1}{2}\right)}{2(m+1)\sin\left(\omega_0\frac{1}{2}\right)}e^{-j\left(\frac{m}{2}\omega_0+\frac{\pi}{2}\right)} \end{aligned} \tag{2.1.15}$$

2.1.3 Computation of A_3

Hopefully the fact that p_3 is the complex conjugate of p_2, $p_3 = p_2^*$, means that $A_3 = A_2^*$.

2.2 Time domain response

The computation of the time domain response is straightforward as was previously stated in (2.1.1):

$$\begin{aligned} y_p(n) &= \sum_{k=1}^{n} A_k\,(p_k)^n\,u(n) \\ &= \left(A_1 + A_2\left(e^{j\omega_0}\right)^n + A_2^*\left(e^{-j\omega_0}\right)^n\right)u(n) \\ &= \left(\left(A_2e^{jn\omega_0}\right) + \left(A_2e^{jn\omega_0}\right)^*\right)u(n) \\ &= 2|A_2|\cos(n\omega_0 + \angle A_2)\,u(n). \end{aligned} \tag{2.2.1}$$

Going back to (2.1.15) we can easily obtain the absolute value $||$ and the phase $\angle$ of A_2:

$$|A_2| = \frac{\sin\left(\omega_0 \frac{m+1}{2}\right)}{2(m+1)\sin\left(\omega_0 \frac{1}{2}\right)} \tag{2.2.2a}$$

$$\underline{/A_2} = -\left(\frac{m}{2}\omega_0 + \frac{\pi}{2}\right), \tag{2.2.2b}$$

provided that (2.2.2a) is positive [6]. In this particular case, we can say that the input (2.0.5)

$$x(n) = \sin(\omega_0 n)\, u(n)$$

has the SSR

$$\begin{aligned} y(n)|_{SSR} &= 2|A_2| \cos(n\omega_0 + \angle A_2)\, u(n) \\ &2|A_2| \sin\left(n\omega_0 + \angle A_2 + \frac{\pi}{2}\right) u(n) \end{aligned} \tag{2.2.3}$$

where we have used

$$\cos(\alpha) = \sin\left(\alpha + \frac{\pi}{2}\right). \tag{2.2.4}$$

3 A general result

Now, we go backward to section 2.1.2 to obtain a general result for A_2 as a function of $H(z)$. We start at (2.1.9):

$$\begin{aligned} A_2 &= Y(z)(1 - e^{j\omega_0} z^{-1})\Big|_{z=e^{j\omega_0}} \\ &= H(z)X(z)(1 - e^{j\omega_0} z^{-1})\Big|_{z=e^{j\omega_0}} && (3.0.1a) \\ &= H(z)\frac{\sin(\omega_0)\, z^{-1}}{(1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1})}(1 - e^{j\omega_0} z^{-1})\Bigg|_{z=e^{j\omega_0}} && (3.0.1b) \\ &= H(z)\frac{\sin(\omega_0)\, z^{-1}}{(1 - e^{-j\omega_0} z^{-1})}\Bigg|_{z=e^{j\omega_0}} \\ &= H(e^{j\omega_0})\frac{\sin(\omega_0)\, e^{-j\omega_0}}{(1 - e^{-j2\omega_0})} && (3.0.1c) \\ &= H(e^{j\omega_0})\frac{\sin(\omega_0)\, e^{-j\omega_0}}{2je^{-j\omega_0}\sin(\omega_0)} && (3.0.1d) \\ &= \frac{1}{2j}H(e^{j\omega_0}), && (3.0.1e) \end{aligned}$$

where we have used (2.0.6) to go from (3.0.1a) to (3.0.1b), and (2.1.14) to go from (3.0.1c) to (3.0.1d).

Now we can rewrite (2.2.2), which is a particular result for a particular $H(z)$, using (3.0.1e) to obtain a general result as a function of $H(z)$:

$$|A_2| = \frac{1}{2}|H(e^{j\omega_0})| \tag{3.0.2a}$$

$$\underline{/A_2} = \underline{/H(e^{j\omega_0})} - \frac{\pi}{2}. \tag{3.0.2b}$$

[6] If not, we change the sign of (2.2.2a) and add π to (2.2.2b).

Finally we can write the SSR of a sinusoidal input of discrete frequency ω_0

$$x(n) = \sin(\omega_0 n)\, u(n)$$

as a function of $H(z)$ evaluated at $z = e^{j\omega_0}$ in the following way:

$$\begin{aligned} y(n)|_{SSR} &= |H(e^{j\omega_0})| \cos\left(n\omega_0 + \angle H(e^{j\omega_0}) - \frac{\pi}{2}\right) u(n) \\ &= |H(e^{j\omega_0})| \sin\left(n\omega_0 + \angle H(e^{j\omega_0})\right) u(n). \end{aligned} \tag{3.0.3}$$

4 Conclusions

We have started computing the response of a particular system to a sinusoidal input. In the process we have discarded the parts of the response that disappear after a certain time, the transient response, to focus on the steady-state response.

Next, we have isolated the part of the response that depends on the system to obtain the general result that says that the response of a LTI discrete-time system, characterized by the transfer function $H(z)$, to a sinusoidal input $x(n)$, with frequency ω_0, amplitude A and phase ϕ, has a sinusoidal SSR $y(n)|_{SSR}$ with the same input frequency ω_0, an amplitude being that of the input A multiplied by the absolute value of $H(z)$ evaluated at $z = e^{j\omega_0}$, and a phase being that of the input ϕ plus the phase of $H(z)$ evaluated at $z = e^{j\omega_0}$.

This statement can be summarized using the following equations. The sinusoidal SSR to the input

$$x(n) = A\cos(n\omega_0 + \phi)\, u(n). \tag{4.0.1}$$

is

$$y(n)|_{SSR} = A|H(e^{j\omega_0})| \cos\left(n\omega_0 + \angle H(e^{j\omega_0}) + \phi\right) u(n). \tag{4.0.2}$$