Sampling of Continuous-Time Signals

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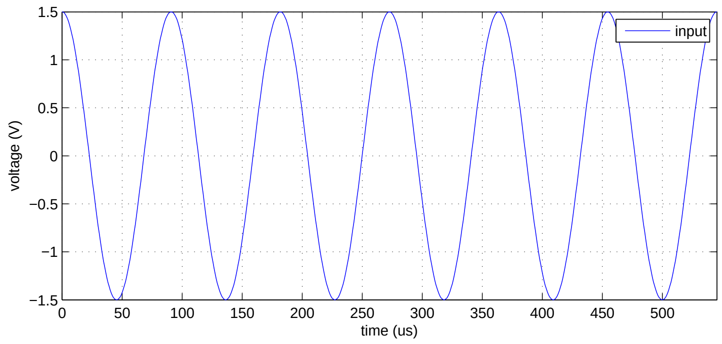
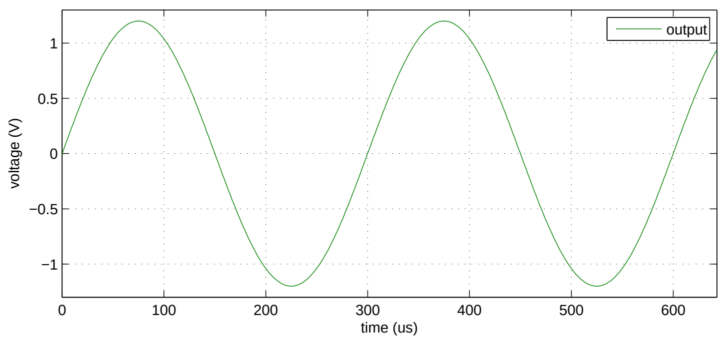
December 12, 2019

1 Subsampling a sinusoidal signal

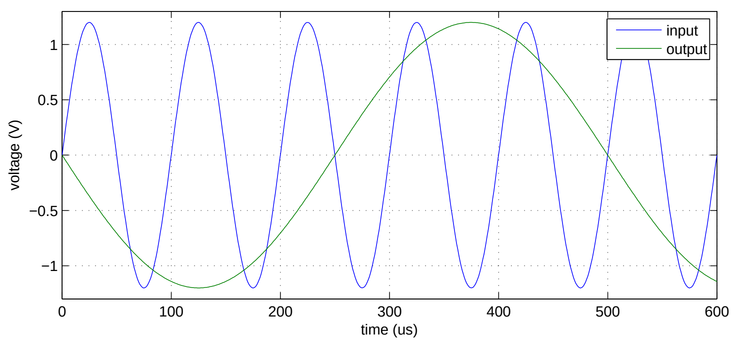
PROBLEM 1.1 Consider sampling the sinusoidal signal of the figure with an ideal (i.e no quantization error) ADC. The samples are send to an ideal (i.e. ideal interpolator) DAC. How the obtained signal at the output of the DAC will be seen in an oscilloscope? Let the sampling rate be $F_s = 10\,\text{kHz}$. Repeat for $F_s = 20\,\text{kHz}$.



PROBLEM 1.2 Consider sampling a sinusoidal signal with an ideal ADC followed by an ideal DAC. Let the sampling rate be $F_s = 7\,\text{kHz}$. The figure shows the obtained signal at the output of the DAC. How the signal at the input of the ADC will be seen in an oscilloscope? Are you sure that this is the only possible signal?

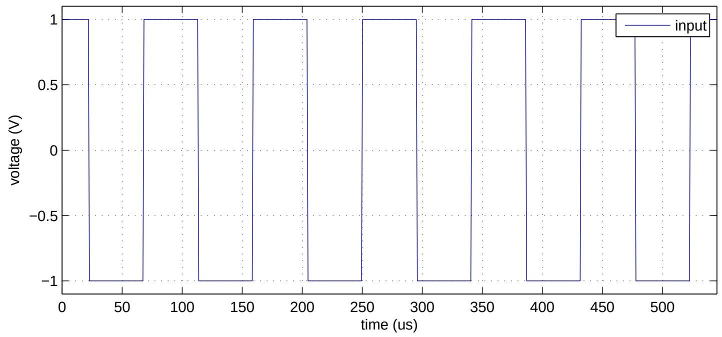


PROBLEM 1.3 Consider the two signals in the figure. The *output* signal is the result of sampling the *input* signal with an ideal ADC followed by an ideal DAC. Determine the sampling rate F_s that has been used. Sure?

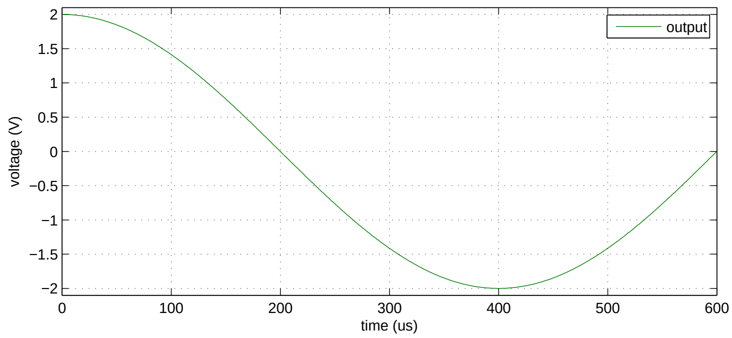


PROBLEM 1.4 Consider the square signal of the figure. This signal is low-pass filtered by an ideal filter in order to remove all the harmonics higher than the fundamental frequency. The low-pass filtered signal is sampled using an ideal ADC followed by an ideal DAC. How the obtained signal at the output of the DAC will be seen in an oscilloscope? Let the sampling rate be $F_s = 10\,\text{kHz}$.

Hint: A square wave with a peak-to-peak amplitude of value A_{pp} has a first harmonic with a peak-to-peak amplitude $A_{pp}\frac{4}{\pi}$.

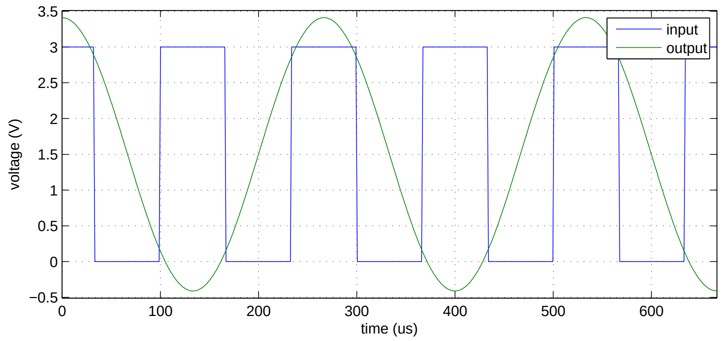


PROBLEM 1.5 Consider a square signal low-pass filtered as before. The low-pass filtered signal is sampled using an ideal ADC followed by an ideal DAC. Let the sampling rate be $F_s = 10\,\text{kHz}$. The figure shows the obtained signal at the output of the DAC. How the square signal will be seen in an oscilloscope? Are you sure that this is the only possible signal?



PROBLEM 1.6 Consider the two signals in the figure. The *output* signal is the result of sampling the, previously low-pass filtered as before, *input* signal with an ideal ADC followed by an ideal DAC. Determine the sampling rate F_s that has been used. Sure?

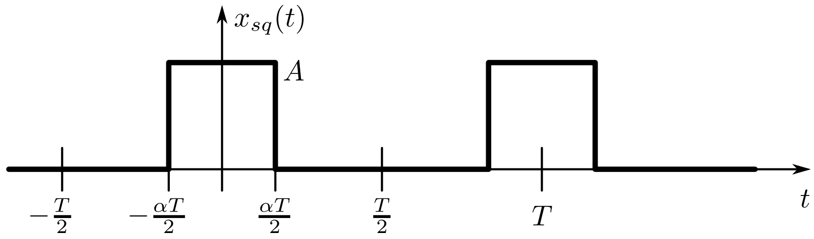
Hint: $\cos(2\pi F_a t) = \cos(2\pi(-F_a)t)$.



2 Subsampling a periodic signal

Non-sinusoidal periodic waveforms are an important class of signals with some prominent examples as the square and sawtooth waveform. The Fourier series method of analysis first resolves a periodic input into the sum of a dc component and infinitely ac components at harmonically related frequencies.

For instance, a rectangular signal like the one in the figure

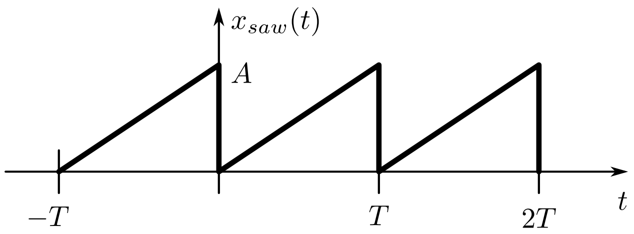


can be resolved into the following sum: $x_{sq}(t) = \alpha A + \frac{2A}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n\alpha\pi)}{n} \cos(n\omega_0 t)$.

The square signal is a particular case of the rectangular signal when $\alpha = 0.5$ (i.e a duty cycle of 50%) and the previous equation may be written as:

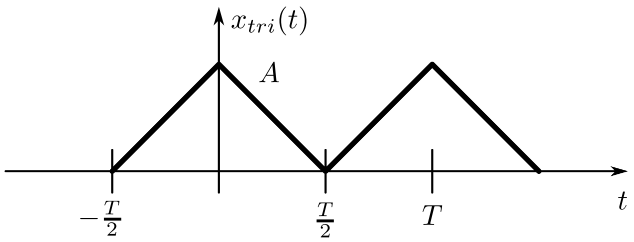
$x_{sq}(t) = \frac{A}{2} + \frac{2A}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cos((2n+1)\omega_0 t) = \frac{A}{2} + \frac{2A}{\pi} \left(\cos(\omega_0 t) - \frac{1}{3}\cos(3\omega_o t) + \frac{1}{5}\cos(5\omega_o t) - \ldots \right)$.

Another example is the sawtooth signal of the figure



that can be resolved into the following sum: $x_{saw}(t) = \frac{A}{2} - \frac{A}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(n\omega_o t)$.

A final example is the triangular of the figure

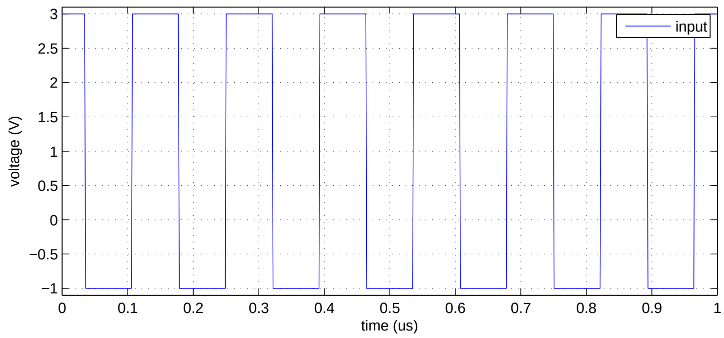


that can be resolved into the following sum:

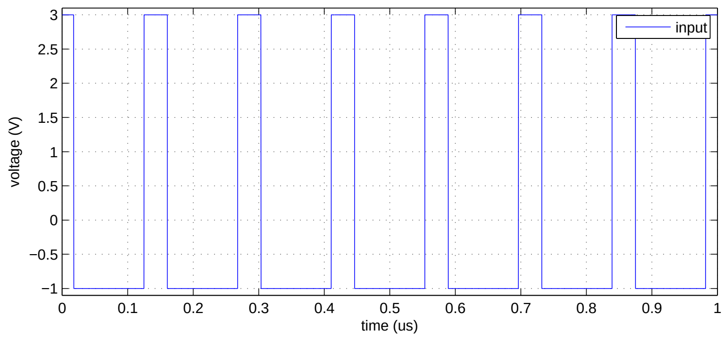
$x_{tri}(t) = \frac{A}{2} + \frac{4A}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos((2n+1)\omega_o t) = \frac{A}{2} + \frac{4A}{\pi^2} \left(\cos(\omega_0 t) + \frac{1}{9}\cos(3\omega_o t) + \frac{1}{25}\cos(5\omega_o t) + \ldots \right)$.

An important difference between the last and the previous waveforms is the way harmonics decrease with frequency. While square and sawtooth components have an amplitude inversely proportional to the frequency (i.e. amplitude at frequency $n\omega_0 \propto \frac{1}{n}$), triangular components have an amplitude inversely proportional to the square of the frequency (i.e. amplitude at frequency $n\omega_0 \propto \frac{1}{n^2}$). This means that a triangular waveform has the spectrum more concentrated at low frequencies. Consequently, a low-pass filter will have much more effect on a square than on a triangular waveform.

Problem 2.1 Consider the square signal of the figure with a 50% duty cycle, i.e. $\alpha = 0.5$. This signal is sampled at $F_s = 20\,\text{MHz}$ using an ideal ADC followed by an ideal DAC. Determine the obtained signal at the output of the DAC as a sum of sinusoidal signals. Draw its positive spectrum made of Dirac deltas at the right frequency and with the right amplitude. Number each delta with a number n corresponding to the nth-harmonic of the square signal at frequency n times the fundamental. Limit your results up to the 9th-harmonic.



Problem 2.2 Repeat the previous problem considering now a rectangular signal with a 25% duty cycle, i.e. $\alpha = 0.25$.



Problem 2.3 Repeat the previous problem (a rectangular signal with a 25% duty cycle, i.e. $\alpha = 0.25$) now with a frequency $F_0 = 95.111\,\text{MHz}$. Let the sampling rate be $F_s = 80\,\text{MHz}$. Limit your results up to the 6th-harmonic.

3 Signal to noise ratio (SNR)

As you know a real ADC quantizes the samples of a signal with a finite number of bits b. As a consequence it introduces an error with such properties that it can be viewed as a certain type of noise. The quantization noise of a signal sampled by an ADC with b bits and a dynamic range D_r can be approximated by $N_q = D_r^2/(12 \cdot 2^{2b})$. The approximation assumes that the sampled signal is a line between two quantized values (which is a reasonable assumption, for any signal, if b is high enough). For a sinusoidal signal of peak amplitude A, the power is $S = A^2/2$. So, the SNR$= S/N_q = \frac{6A^2}{D_r^2} 2^{2b}$. When the signal uses all the dynamic range, i.e. the peak to peak amplitude equals the dynamic range $2A = D_r$, the SNR has its maximum value: SNR$= S/N_q = 1.5 \cdot 2^{2b}$ or SNR$|_{dB}= 10\log_{10}(S/N_q) = 10\log_{10}(1.5) + 10\log_{10}(2^{2b}) = 10\log_{10}(1.5) + 20\log_{10}(2)b$ that can be approximately written as SNR$|_{dB}\simeq 1.76 + 6.02b$.

Problem 3.1 Consider an ADC of $b = 8$ bits an dynamic range of $D_r = 5\,\mathrm{V}$. Compute the SNR after the discretization and quantification of a sinusoidal signal. Let the peak amplitude be $A = 1.25\,\mathrm{V}$. Compare this result with the one obtained when the input signal uses all the dynamic range of the converter, i.e. $A = 2.5\,\mathrm{V}$.

Problem 3.2 Consider an ADC with a dynamic range of $D_r = 5\,\mathrm{V}$ that discretizes and quantifies a sinusoidal signal with peak amplitude A. Let the peak amplitude be $A_0 = 2.5\,\mathrm{V}$, $A_1 = 1.25\,\mathrm{V}$ and $A_2 = 0.625\,\mathrm{V}$. For each amplitude, how should b be chosen so that the SNR$> 49\,\mathrm{dB}$.

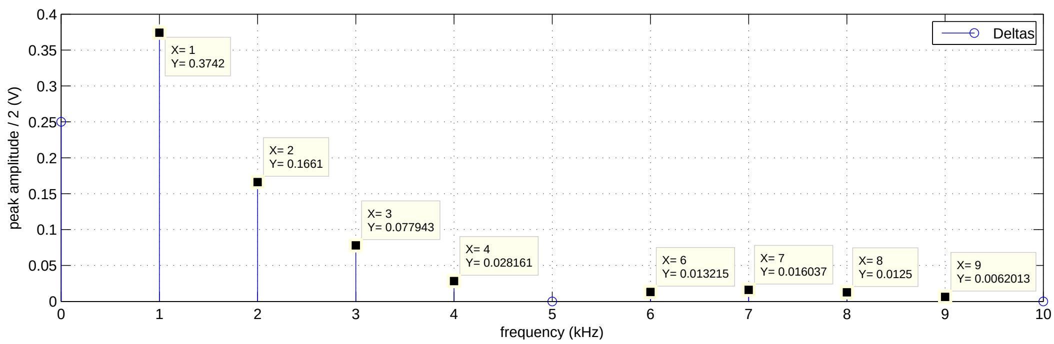
4 Total harmonic distortion (THD)

In the real world there are no ideal devices. In particular there are no linear devices. The nonlinearity of same devices is characterized putting a pure sinusoidal signal at the input. A linear device will have a pure sinusoidal signal at the *fundamental* frequency at the output. A nonlinear device will have harmonics (if you are interested ask for an explanation) at multiples of the *fundamental* frequency at the output. The nonlinearity is evaluated as the ratio of the power of the signal at the *fundamental* frequency to the power of all other harmonics (excluding the dc component). This relation is called Total Harmonic Distortion (THD). We can express this relation as THD= S/D, being S the power of the signal at the *fundamental* frequency and D the power of all other harmonics. Alternatively we can express this ration in dB as $\text{THD}_{dB} = 10\log_{10}(S/D)$.

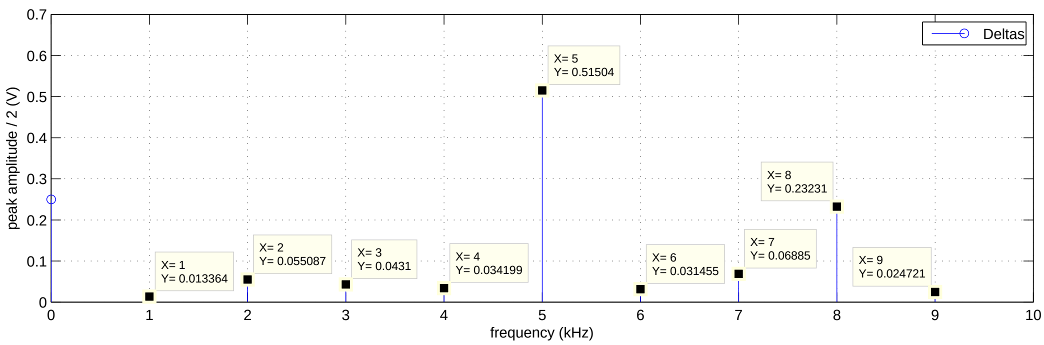
Harmonic distortion is also observed in ADC, with the particularity that once the sinusoidal signal is sampled the relations between the harmonics could not be the expected because of the subsampling phenomenon previously worked, e.g. in *Problem* 2.2 or *Problem* 2.3.

PROBLEM 4.1 Consider a sinusoidal signal of frequency $F_0 = 1\,\text{kHz}$ at the input of a nonlinear ADC. The obtained spectrum frequency (via Fourier Transform) at the output of the ADC is the one showed below. Compute the THD in dB.

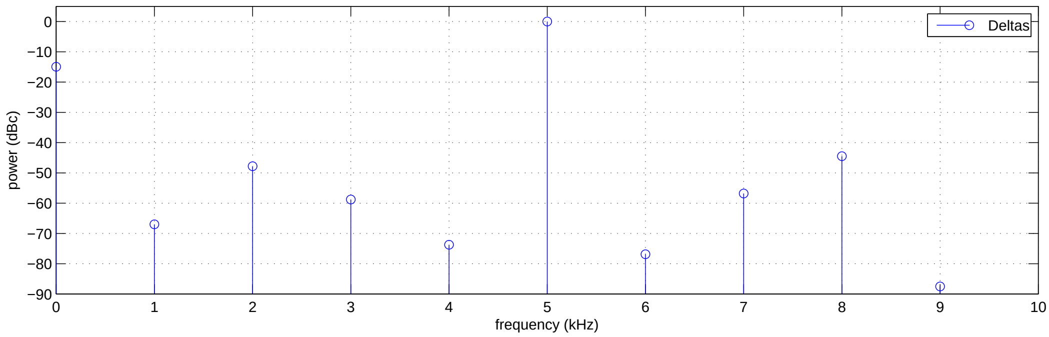
Hint: don't forget the part of the spectrum with negative frequencies, i.e the sinusoidal signal at frequency 1 kHz has two deltas and its power is computed from both.



PROBLEM 4.2 Consider a sinusoidal signal of frequency $F_0 = 5\,\text{kHz}$ at the input of a nonlinear ADC. The obtained spectrum at the output of the ADC is the one showed below. Note that, as the sampling frequency is $F_s = 18\,\text{kHz}$, subsampling phenomenon is observed. Compute the THD in dB.



PROBLEM 4.3 Consider a sinusoidal signal of frequency $F_0 = 5\,\mathrm{kHz}$ at the input of a nonlinear ADC. The obtained spectrum in dB with respect to the fundamental at the output of the ADC is the one showed below. Note that, as the sampling frequency is $F_s = 18\,\mathrm{kHz}$, subsampling phenomenon is observed. Compute the THD in dB.



PROBLEM 4.4 Consider that the spectrum computed in *Problem* 2.2 is the result of sampling a pure sinusoidal signal and not a rectangular signal. This way, you should consider that the harmonics comes from distortion of the ADC. Compute the THD in dB.

PROBLEM 4.5 Consider that the spectrum computed in *Problem* 2.3 is the result of sampling a pure sinusoidal signal and not a rectangular signal. This way, you should consider that the harmonics comes from distortion of the ADC. Compute the THD in dB.

5 Signal to noise and distortion ratio (SINAD)

In previous sections we have seen that an ADC adds noise and distortion to the sampled signal. The first is because of the quantization process and the second because of the nonlinearities of the ADC. We have characterized the effect of noise as the ratio of the signal power to the noise power (SNR) and the effect of distortion the ratio of the signal power to the distortion power (THD).

Now we will consider the effect of noise and distortion simultaneously as the ratio of the signal power to the noise plus distortion power, called signal to noise and distortion ratio (SINAD). We can express this relation as SINAD= $S/(N+D)$, being S the power of the signal at the *fundamental* frequency, N the power of the quantization noise and D the power of all other harmonics. Alternatively we can express this ration in dB as $\text{SINAD}_{dB} = 10\log_{10}(S/(N+D))$.

PROBLEM 5.1 Compute the SINAD in dB of the *Problem* 4.2 considering that the number of bits used to code the samples is $b = 4$ and the D_r the necessary to fit the input signal. How is the noise compared to the distortion? Do we have any improvement when increasing the number of bits to $b = 5$? Compare the values of SNR, THD and SINAD and give some conclusions.

PROBLEM 5.2 Compute the SINAD in dB of the *Problem* 4.3 considering that the number of bits used to code the samples is $b = 4$ and the D_r the necessary to fit the input signal. How is the noise compared to the distortion? Do we have any improvement when increasing the number of bits to $b = 5$? Compare the values of SNR, THD and SINAD and give some conclusions.

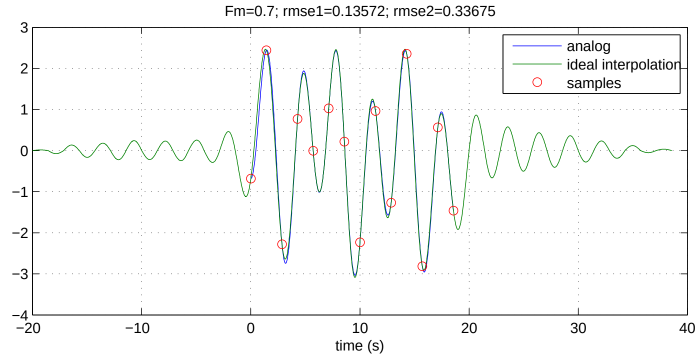
6 Effective Number of Bits (ENOB)

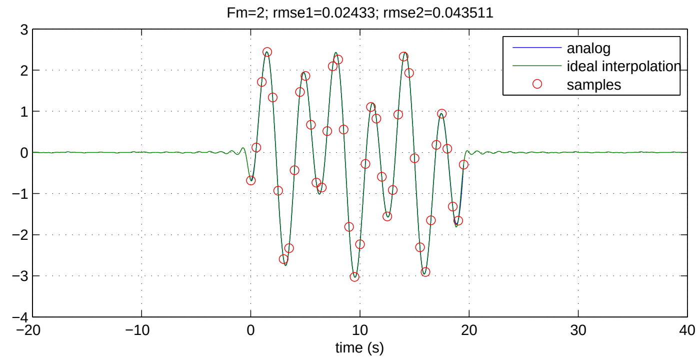
On *Section* 3 we have obtained the following relation: $\mathrm{SNR}|_{dB} \simeq 1.76 + 6.02b$. If in addition to noise we want to take into account the distortion, and see it as another source of noise, we define and effective number of bits (ENOB) as the number of bits that will give and SNR equal to the SINAD: $\mathrm{SINAD}|_{dB} \simeq 1.76 + 6.02\mathrm{ENOB}$. Once the value of SINAD is known the ENOB can be computed from the previous equation. Note that ENOB will always be lower that the number of bits b. ENOB is a simple way to summarize the overall performance of an ADC.

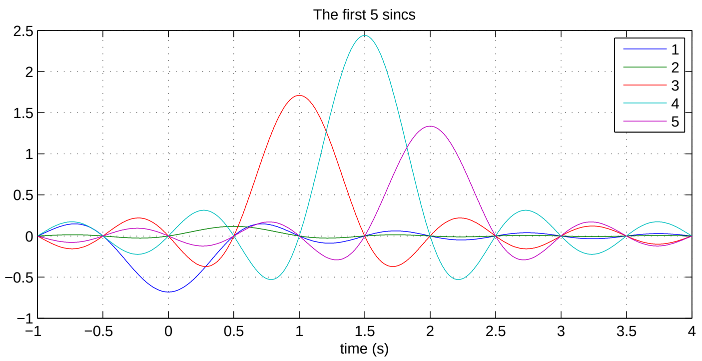
PROBLEM 6.1 Compute the ENOB of *Problem* 5.2.

7 The ideal interpolator

If you are interested in knowing more about the ideal band-limited interpolator ask your teacher.







8 ADC vs Downconverters

Read pages 10 and 11 of the following article *October 2014 Edition of EE Times Europe*. Below you will find these two pages.

PROBLEM 8.1 Summarize the article and relate it to the *Digital Signal Processing* course you are following.

Analog-to-digital converters are continuing to reach higher frequencies at which signals captured over the air can be directly digitized – eliminating microwave downconverters in the process. Total downconverter elimination may be inevitable – but it won't occur soon.

Will Direct-to-Digital Conversion Displace Microwave Converters?

By Barry Manz for Mouser Electronics

IMAGINE WHAT LIFE would be like without the ability to convert analog information to the digital domain. The world as we know it might be like the 1950s, with no smartphones, HDTV, computers, or smartwatches. The list of the things that either would not exist or would be hobbled without analog-to-digital conversion (ADC) would be enormous.

Most of the items on that list require ADC at low frequencies (and with digital-to-analog conversion, "DAC"). In the higher microwave region of the electromagnetic spectrum, conversion is still elusive. The benefits of conversion at very high frequencies, presently impossible, are cause for significant research by industry, academia, and defense. A great potential benefit is the ability to eliminate microwave downconverters and a significant number of analog components.

The task of ingesting and digitizing higher instantaneous bandwidths falls to ADCs, which already possess formidable function. On the face of it, converting analog waveforms to the digital domain is no more complex at high frequencies. Conversions at high sampling rates, resolution, and dynamic range have long been a fait accompli.

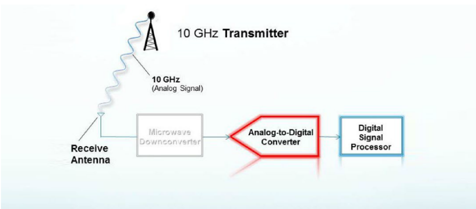


Figure 1. A 10 GHz radio signal is picked up by the receive antenna. The microwave downconverter reduces the frequency to one that the system is able to process. The ADC converts it to 1s and 0s. As a digital signal, it is now possible to perform many functions that are not as easily done in analog format. But if the ADC can directly ingest the original 10 GHz analog signal, you eliminate the need for a big and expensive microwave downconverter.

It's Complicated

Moore's Law doesn't apply to converters; leaps forward in sampling rate (and thus bandwidth) are extremely difficult to achieve. The most potent devices reside in the labs of test equipment manufacturers, device vendors, and the military. Just how high a frequency achieved to date is not clear, and there's good reason for holding state-of-the-art conversion secrets closely. Converters are key performance determinants for oscilloscopes, spectrum and signal analyzers, and virtually every defense electronic system that relies on the electromagnetic spectrum. Running a close second are FPGAs, such as Altera's Stratix V GX, and more recently, general-purpose graphics processing units that process massive amounts of data resulting from high-frequency direct-to-digital conversion.

A simplified example: an ADC sampling at 20 Gb/s, which makes its instantaneous bandwidth about 10 GHz when operated in the first Nyquist zone. Assuming that the device has an Effective Number of Bits (ENOB) of 10, and a dynamic range of 70dB, it would be possible for the ADC to directly ingest DC to 10GHz with excellent resolution and dynamic range, without interstitial microwave downconverters to reduce the frequency to one that the ADC can handle.

The result would be a major reduction in hardware, complexity, size, weight, and cost, and a commensurate increase in speed and flexibility; critical in many defense applications. One such example is the truly frightening interaction between search-and-fire control radars and the radar warning receivers (RWRs) and jammers in fighter aircraft. In a typical scenario, a radar acquires the target (fighter jet). The target's jamming radar must capture the signal, digitize it, and then retransmit it as a waveform that essentially confuses the sourcing radar within less than a second.

In both the fighter's RWR and Electronic Warfare systems, and in the adversary's radar, the ADC is the initial component determining frequency, speed, and degree of fidelity in signal acquisition. High-speed signal processing, the next element in the chain, has the daunting task of processing and transforming it into waveforms that can confuse the adversarial radar. The DAC must reconvert data from digital to analog, after which it is retransmitted at or near the original frequency. If the ADC and DAC cannot directly capture the signal, downconversion and upconversion are required, which is why direct-to-digital conversion is so desirable.

Benefits are equally applicable to terrestrial or satellite communications, test and measurement, spectrum monitoring, and virtually any system operating at microwave frequencies. Today, that includes wireless systems ranging from carrier-based networks to public safety, Wi-Fi, machine-to-machine communication, distributed antenna systems, and more.

A Long Road

The microwave industry has not completely embraced what is almost certainly the way microwave systems will be constructed in the future. One reason for inaction is the long standing tension between digital and analog designers, who have rarely crossed boundaries. Additionally, the microwave industry is historically slow to change (except semiconductor suppliers), and change that does occur has mostly originated from military customers. At present, manufacturers of microwave downconverters are relatively safe in designing products at frequencies above ~5GHz where the combination of sample rate, resolution, and dynamic range are sufficient to provide improved performance for most applications.

Still, converter technology is continuously moving forward in sampling rate, instantaneous bandwidth, resolution, and dynamic range. Analog Devices' AD9250 series (Figure 2) combines high sampling rate, resolution, and dynamic range for demanding systems. Great advances in sampling rate have potential to eliminate chunks of hardware. Microwave components may still be required between antenna and ADC, but quantities will be significantly reduced with an irresistible combination of savings in reaction time, space, weight, and total system cost.

Figure 2. The Analog Devices AD9250 ADC has all the attributes required for precision conversion.

A Complex Mixture

It's tempting to focus on sampling rate and instantaneous bandwidth as benchmarks defining converter performance but they are meaningless without including others like effective number of bits (ENOB) and spurious-free dynamic range. But design trade-offs are such that the highest sampling rate rarely occurs in the same device with the widest bandwidth and dynamic range. For example, converters have been demonstrated with sampling rates of a stunning 64 Gsamples/s, so instantaneous bandwidth might be half as much (32GHz.) Such a device, if producible, would be breathtaking and could effectively eliminate analog downconversion up to 32GHz, well into the millimeter-wave region. However, the stated ENOB would probably be 5.8 at 10GHz, so at higher frequencies it could drop off dramatically, and dynamic range could be about 43dB. As a laboratory device, these values are impressive. However, dynamic range and ENOB would need to increase before it could be useful, and they ultimately will.

With all converters, bits of resolution largely influence the degree to which the device can accurately represent the input signal. ENOB is essentially a simple way to summarize the overall performance of an ADC and particularly its accuracy at a given frequency and sampling rate. Unfortunately, as frequency increases, ENOB decreases, because various noise and distortion components also follow this curve, reducing signal-to-noise ratio and thus ENOB (as these metrics are intrinsically related). The term ENOB can become confusing because it can refer to the bits of resolution achievable by the ADC and the total "effective" number of bits that it achieves when part of a complete system. Device ENOB is invariably higher than when it is included in a system. For this discussion, it makes sense to use the device-level number.

In conversion devices dynamic range is the breadth of signal amplitudes in decibels that the device can resolve, making it an essential metric for communication, EW, radar, and other applications in which signal strength varies rapidly and often over a huge range. Higher numbers are better. Together, ENOB, dynamic range, sampling rate, and bandwidth tell much about performance. Performance at high input frequencies and in demanding applications determines whether a signal can be identified in noisy, dense spectral environments and if characteristics can be identified with precision.

Summary

Does direct-to-digital conversion put an end to microwave downconverters? It has already done away with them at lower microwave frequencies. However, as microwave frequencies rise it will probably take many years before direct conversion at, say, 40GHz will be possible while preserving key characteristics. And conversion is not the only consideration. High rates of conversion produce large amounts of data very quickly and would require "big-data" processing power, requiring more hardware, space, and power than is practical for average users. Someday these feats will be attainable.

For more information, visit the section of the Mouser website Applications & Technologies.